**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part II. Optimization methods

## Lecture 12. Optimization control problem with fixed final state

We considered optimization control problems with free final state. However there exist the problems with fixed final case. We would like to shift the system from a given initial state to the given final step with minimization some functional in this case. We will determine the necessary conditions of the optimality for these problems. The time optimal problem will be considered as an example. The shooting method will be used as a practical algorithm for solving necessary conditions of the optimality.

### 12.1. Problem Statement

Consider optimization control problems with fixed final state. We have the system described by differential equation

  (12.1)

with initial condition

 *х*(0) = *х*0.(12.2)

Besides it is given the final state

 *х*(*Т*) = *х*1.(12.3)

The set of the admissible controls



We have the integral



The smooth enough functions *f* and *g*, the functions *a* and *b*, and the numbers *T*, *х*0and *х*1are given here.

**Problem 12.1.***Minimize the integral I on the set U, where x satisfies the equality* (12.1) – (12.3).

### 12.2. Optimality conditions

We will use the known technique for the analysis of the given problem. Determine Lagrange function



It is equal to *I* if the function *x* is the solution of the equation (12.1). Let *u* is a solution of the Problem 12.1, and *x* is the corresponding state function. We get the inequality

Δ*I* = *I*(*v*) – *I*(*u*) ≥ 0 ∀*v*∈*U.*

Then we have

Δ*L* = *L*(*v*,*y*,*р*) – *L*(*u*,*x*,*р*) ≥ 0 ∀*v*∈*U*, ∀*р*.

where *y* is the solution of the system (12.1) – (12.3) for the control *v.* Find the value



where

 *H*(*t*,*u*,*x*,*р*) = *р f*(*t*,*u*,*x*) – *g*(*t*,*u*,*x*), (12.4)

Δ*H* = *H*(*t*,*v*,*y*,*р*) – *H*(*t*,*u*,*x*,*р*).

Using the standard technique we get the inequality



where

Δ*uH* = *H*(*t*,*v*,*x*,*р*) – *H*(*t*,*u*,*x*,*р*),

and *η* is the high term. After the integration by parts we obtain



because of the conditions (12.2) and (12.3). Hence we get



Let the function *p* be the solution of the adjoint equation

 . (12.5)

So we have the inequality



Using the standard technique we get the maximum principle

  (12.6)

**Theorem 12.1.** *The solution u of Problem* 12.1 *satisfies the maximum principle* (12.6), where the functions *x* and *p* are the solutions of the system (12.1), (12.2), (12.3), (12.5).

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| **Question**: *The adjoint equation does not have any additional conditions. What is its sense?* |

Note the difference between this result and the optimality conditions for the optimization control problem with free final state. We had before state equation with initial condition and adjoint equation with final condition. Now we have initial and final conditions for the state equation, and we do not have any conditions for the adjoint equation. However we have two differential equations with two boundary conditions. Indeed we can analyze the maximum principle (12.6) as at the optimization problem with free final state. Hence we determine a dependence of its solution from the state and adjoint functions

  (12.7)

where Ф is a concrete function. Then we put it to the state and adjoint equations. We get the system of two differential equations with respect to the state and adjoint functions with two boundary conditions. So our optimality conditions have the sense in principle.

Table 12.1. The algorithm of solving for optimization problem with fixed final state.

|  |  |  |
| --- | --- | --- |
| **step** | **action** | **remark** |
| 1 | Definition of the concrete values of known functions and parameters | The concrete problem is transformed to the standard form.  |
| 2 | Definition of the function *H*. | Using the formula (12.4). |
| 3 | Definition of the adjoint equation. | Using the formula (12.5). |
| 4 | Solving the maximum principle.  | Determination of the dependence (12.6) of the control from the state and adjoint functions from (12.6) |
| 5 | Solving the state and adjoint equations. | Finding of the general solution of the system (12.1), (12.5) after the substitution here the solution of the maximum principle. |
| 6 | Finding the state and adjoint functions. | Using the boundary conditions (12.2), (12.3) for finding the state and adjoint functions. |
| 7 | Finding the solution of the necessary conditions of the optimality. | Substitution the state and adjoint functions to the formula of the control from the step 4. |
| 8 | Analysis of the result. | This control can be optimal, but it can be non-optimal too.  |

### 12.3. Practical solution of the necessary conditions of the optimality

We obtain the system of the optimality conditions (12.1), (12.2), (12.3), (12.5), (12.6). Then we have the necessity to have the constructive method of its solving.

|  |
| --- |
| **Question**: *Could we use the known iterative method for solving this system?* |

We cannot to use the known iterative method for solving this system because we do not have any conditions for the adjoint equation, and we have two conditions for the state equation. So we will use other algorithm. It is based on the *shooting method*.

Determine the fictitious initial state for the adjoint equation

  (12.8)

where the number *α* is unknown now. We get system (12.1), (12.2), (12.3), (12.5), (12.6), (12.8) with respect to three unknown functions and parameter *α*. Then the system (12.1), (12.2), (12.5), (12.8) is Cauchy problem with respect to the state and adjoint functions. Its solutions depend from *α* of course. Now the equality (12.3) can be interpreted as an equation with respect to *α*

  (12.9)

where  and  is the solution of the system (12.1), (12.2), (12.3), (12.5), (12.6), (12.8) for the concrete value *α*.

We have the algebraic equation (12.9) with respect to the number *α*. It can be solved, for example by iterative method

  (12.10)

where  is a parameter of the algorithm. So we have the following algorithm. At first we choose an initial approximation of the control *u*0 and *α*0of parameter*α*. If the control *uk* and parameter *αk* are given, then the functions  and  are determined from Cauchy problem (12.1), (12.2), (12.5), (12.8) with known value  Then we find the next iteration  by the formula



because of the equality (12.7). The value  is determined by the formula



If the sequence  converges, then its limit is the solution of the necessary conditions of the optimality.

### 12.4. Example

Consider the system

  (12.11)

 . (12.12)

It is given the set of the admissible controls



and the integral



We have the problem of the minimization the value *I* on the set *U* with additional final condition

 . (12.13)

Determine the function

*H = р u* – (*u*2 + *x*2)/2.

We obtain the adjoint equation

  (12.14)

Find the control from the maximum principle

  (12.15)

So we have the system (12.11) – (12.15) with respect to three unknown functions *u*, *x*, and *p.* Determine the algorithm of solving this system.

1. Choose the sequence  and initial approximations *u*0 and *α*0.
2. For the known values *uk* and *αk* find the solution of Cauchy problem





1. Find next iteration of the control by the formula



1. Find next iteration of the parameter by the formula



### 12.5. Vector case

Consider control system described by the differential equations

  (12.16)

with the initial conditions

 *х*(0) = *х*0(12.17)

and final conditions

 *х*(*Т*) = *х*1,(12.18)

where  is the state vector-function,  is vector control, *f* is the given vector-function, the vector  is the given initial state,  is the given final state. The control *u* is chosen from the set

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where  is the given closed subset of the *r*-dimensional Euclid space. Consider the value



where *g* is a given function.

**Problem 12.2**. *Find the function u = u*(*t*) *from the set U*, *which minimize the value I*.

We can solve this problem with using the standard technique. Determine the function

 **  (12.19)

Then the optimal control satisfies the maximum principle

  (12.20)

where  is adjoint vector-function. It satisfies the adjoint system

  (12.21)

where  The system (2.16), (2.17), (2.18), (12.20), (2.21) can be solved with using shooting method.

|  |
| --- |
| **Question**: *How we can solve the problem, where the part of the final states is fixed?* |

Consider the system



with the initial conditions



and final condition



We have the minimizing functional



and the set of admissible controls



We determine the function

**

The adjoint system consists of the equations



with unique final condition



### 12.6. Time optimization problem

We consider the movement of the body by a force. This phenomenon is described by the Newton’s law



It can be transform to



where the acceleration *u* is the control. We know the initial state *a* and initial velocity *v* of the body. So we have the initial conditions

 

We would like to move the body to the given final state at the final time *T* and to stop it here. We choose the origin of coordinate as the final state. Then we have the final conditions



Our control *u* = *u*(*t*) satisfies the inequality

| *u*(*t*) | ≤ 1.

We would like to choose the admissible control such that the time of the stopping *T* will be minimal.

We transform our problem to the standard form. Determine the state functions



Then we have the system

  (12.22)

with initial conditions

  (12.23)

and final conditions

  (12.24)

The minimizing functional can be transformed to the integral



We determine the function *H* by the formula



The adjoint system is described by the differential equations

  (12.25)

Find the maximum of the function *H* with respect to the control on the interval [-1,1]. This function is linear. Therefore, maximum can be obtained on the boundary this interval only. It depends from the sign of the function *p*2. Hence, the solution of the maximum principle is

  (12.26)

We have the system (12.22) – (12.26) for finding five unknown functions. There are the control, two state functions, and two Lagrange multipliers. Find  from the first equation (12.16), where  is constant. Then we obtain general solution of the second equation (12.26)  where  is constant. Therefore, the function *p*2 is linear. This function can have the constant sign, or it can change the sign one time only. Therefore, we can have four different cases:

i)  ii)  iii)  iv) 

Consider these cases.

If  then we find the solution of the system (12.22), (12.23)



Put it to the equalities (12.24); we get



Then we find  This result can be applicable, if the value *v* is negative, and 

If  then we find the solution of the system (12.22), (12.23)



Put it to the equalities (12.24); we get



Then we find  This result can be applicable, if the value *v* is positive, and 

For the third case we have



for  Then we find

  (2.27)

The second equation (12.22) for  has the general solution Using the second equality (2.27) we find the constant  Therefore, we have



Then the first equation (12.22) for  has the general solution  Using the first equality (2.27) we find the constant  Therefore, we get



Using the equality (2.24) we obtain



We have  from second equality. So the first equality can be transformed to  Then we find the moment of the control switching . The time of the movement *T* satisfies the equality



So we find the final time



This result is applicable, if 

For the fourth case we have



for  Then we find

  (2.28)

The second equation (12.22) for  has the general solution Using the second equality (2.28), we find the constant  Therefore, we have



Then the first equation (12.22) for  has the general solution  Using the first equality (2.28) we find the constant  Therefore, we get



Using the equality (2.24) we obtain



We have  from second equality. So the first equality can be transformed to  Then we find the moment of the control switching . The time of the movement satisfies the equality



So we find the final time



This result is applicable, if 

**Outcome**

* The maximum principle is applicable for the optimization control problems with fixed final state.
* Necessary conditions of the optimality for these problems consist of the state equation with two boundary conditions, the adjoint equation without any boundary conditions, and maximum principle.
* Necessary conditions of the optimality for these problems can be solved with using shooting method.
* These results are applicable for the vector case.
* The time optimal problem can be considered as an application of this theory.

### Task. Optimization control problem for a systems with fixed final time

Consider the control system described by the differential equations



with initial conditions



and final condition

  (\*)

or

  (\*\*)

The set of the admissible control is described by the inequalities



We have the problem of the minimization of the value

.

Table of the parameters.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variants |  |  |  | final condition |
| 1 |  |  |  | (\*) |
| 2 |  |  |  | (\*\*) |
| 3 |  |  |  | (\*) |
| 4 |  |  |  | (\*\*) |
| 5 |  |  |  | (\*) |
| 6 |  |  |  | (\*\*) |
| 7 |  |  |  | (\*) |
| 8 |  |  |  | (\*\*) |
| 9 |  |  |  | (\*) |
| 10 |  |  |  | (\*\*) |
| 11 |  |  |  | (\*) |
| 12 |  |  |  | (\*\*) |
| 13 |  |  |  | (\*) |
| 14 |  |  |  | (\*\*) |
| 15 |  |  |  | (\*) |
| 16 |  |  |  | (\*\*) |
| 17 |  |  |  | (\*) |
| 18 |  |  |  | (\*\*) |

Steps of the task.

1. Write the concrete problem statement.
2. Determine the function *Н.*
3. Determine the adjoint system.
4. Determine the maximum principle.
5. Find the control from the maximum principle.
6. Write the iterative method for solving the conditions of the optimality.

### Next step

We consider the methods of solving extremum problems, which is based on the necessary conditions of the extremum. However these methods transform the initial problem to another problem. There is the system of optimality conditions. Then we used iterative methods for solving this system. Our next step is the analysis of the direct iterative methods of solving extremum problems without using conditions of the extremum.

### Reference

1. Алексеев В.М., Тихомиров В.М., Фомин С.В. Оптимальное управление. – М., Наука, 1979. –
С. 84-93; 103-107.
2. Miersemann E. Calculus of Variations. Lecture Notes. – Leipzig, 2012. – P. 170-174.